Square root of a complex number

Suppose $a+ib$ be a complex no. with square root $x+iy$

ie $\sqrt{a+ib} = x+iy$ where, x & $y \in R$ --- (1)

using (1) $a+ib = x^2-y^2+2ixy$

equating real & img. parts of above eqn. we get

$$\left.\begin{aligned} a &= x^2-y^2 \\ \& \; b &= 2xy \end{aligned}\right\} \quad (A)$$

We know $(x^2+y^2)^2 = x^4+y^4+2x^2y^2$

$$= x^4+y^4-2x^2y^2+4x^2y^2$$

$$= (x^2-y^2)^2+4x^2y^2$$

also $x^2+y^2 = \sqrt{a^2+b^2}$ using (A) —— (B)

adding (A) & (B)

$$2x^2 = a+\sqrt{a^2+b^2}$$

$$\boxed{x = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}}$$

Find the square root of

(i) $3-4i$

(ii) $3+4\sqrt{7}i$ (iii) $-4-3i$

Sol^n: (i) Let $\sqrt{3-4i} = x+iy$ ---- (1)

squarring b.s. we get

$$3-4i = (x^2-y^2) + 2ixy \quad \text{---- (2)}$$

Equating real & img. parts

$$\left.\begin{array}{l} x^2-y^2 = 3 \quad (i) \\ \& \ 2xy = -4 \quad (ii) \end{array}\right\} \text{— (A)}$$

Also we know that

$$(x^2+y^2) = \sqrt{(x^2-y^2)^2 + 4x^2y^2}$$

$$= \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\therefore x^2+y^2 = 5 \quad \text{------ (B)}$$

from (A) & (B) on adding we get

$$\begin{array}{l} x^2+y^2 = 5 \\ \underline{x^2-y^2 = 3} \\ 2x^2 = 8 \end{array} \Rightarrow x^2 = 4 \Rightarrow \boxed{x = \pm 2}$$

put in (A)(ii) $2\times(\pm 2)\, y = -4 \Rightarrow y = \frac{-4}{\pm 4} = \mp$

$$\boxed{y = \mp i}$$

Hence $\sqrt{3-4i} = \pm 2 \mp i = \pm(2-i)$

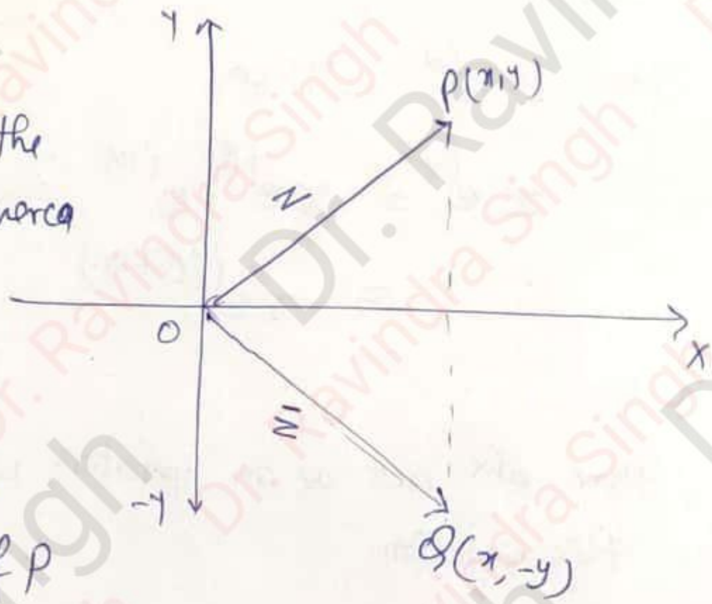
(or xy have -ve sign & $y = \mp$)

Vector Representation of a Complex nos.

Any vector can be obtained by translating the vector in complex plane.

To represent a complex no. $z = x+iy = (x,y)$ by direct line segment or vector $\vec{OP}$ from the plane.

The modulus & argument of the complex no. give the magnitude & direction of the corresponding vector & vice-versa



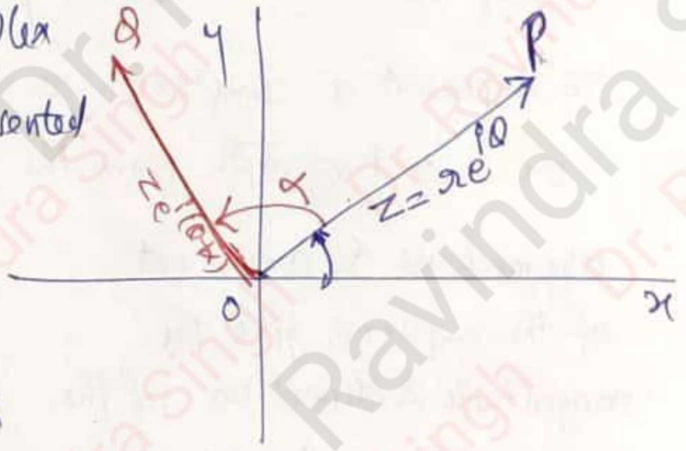
Its' Conjugate. is represented by the reflection or image Q of P in real x-axis

If (r, θ) are the polar coordinates of complex plane P then the polar coordinates of conjugate point Q are $(r, -\theta)$

So that $\boxed{\arg z = -\arg \bar{z}}$

\# If z is a complex no. (vector), interpret geometrically $ze^{i\alpha}$ where α is real.

Sol[n]: Exponential form of a complex no. is, $z = re^{i\theta}$, represented by a vector $\vec{OP}$.



Then $ze^{i\alpha}$ will be

$$ze^{i\alpha} = re^{i\theta} \cdot e^{i\alpha}$$

$$= re^{i(\theta+\alpha)}$$

is also a vector represented by $\vec{OQ}$

Here $e^{i\alpha}$ acts as an operator, which acts on z to produce this rotation.

De Moivre's Theorem

if $z_1, z_2 \cdots$ are the complex nos. s.t.

$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

& $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore z_1 \cdot z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ ---- (1)

or $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$ ---- (2)

Generally if we take the product then (1) goes to

$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n [\cos(\theta_1+\theta_2+\cdots\theta_n) + i\sin(\theta_1+\theta_2+\cdots\theta_n)]$ ---- (3)

if $z_1 = z_2 = z_3 = \cdots$ [redacted] z

then (3) reduce to

$z^n = [r(\cos\theta + i\sin\theta)]^n$

$$\boxed{z^n = r^n[\cos n\theta + i\sin n\theta]}$$

→ This called De Moivre's Theorem.

Roots of a Complex no.

if $\omega^n = z$ then ω is called an n^{th} root of a complex no. z.

also $\omega = z^{1/n}$. If n is a +ve integer then using De Moivre's Theorem we can write

$z^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n}$

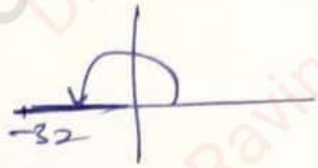
$$\boxed{z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]}$$

When $k = 0, 1, 2, 3 \cdots n-1$

provided $z \neq 0$.

Q. Solve $z^5+32=0$

Sol: $z^5+32=0$



$z^5=-32$

$\therefore -32+0i = r(\cos\theta+i\sin\theta)$

$r=32, \quad \theta=\pi$

$\therefore -32 = 32(\cos\pi+i\sin\pi)$

$\therefore (-32)^{\frac{1}{5}} = \{32(\cos\pi+i\sin\pi)\}^{1/5}$

$= 2\left[\cos\left(\frac{\pi+2\pi k}{5}\right)+i\sin\left(\frac{\pi+2\pi k}{5}\right)\right]$

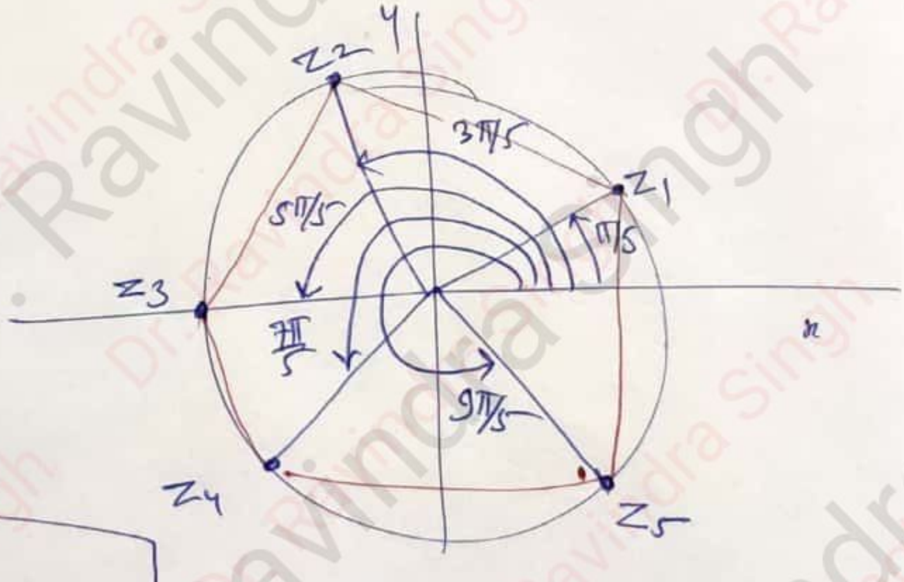
When, $k=0$, $z_1 = 2[\cos\pi/5+i\sin\pi/5]$

When $k=1$, $z_2 = 2[\cos 3\pi/5+i\sin 3\pi/5]$

When $k=2$, $z_3 = 2[\cos 5\pi/5+i\sin 5\pi/5]$

When $k=3$, $z_4 = 2[\cos 7\pi/5+i\sin 7\pi/5]$

When $k=4$, $z_5 = 2[\cos 9\pi/5+i\sin 9\pi/5]$



it makes a regular Pentagon

Find the different roots of $(1+i)^{1/3}$ & locate graphically.

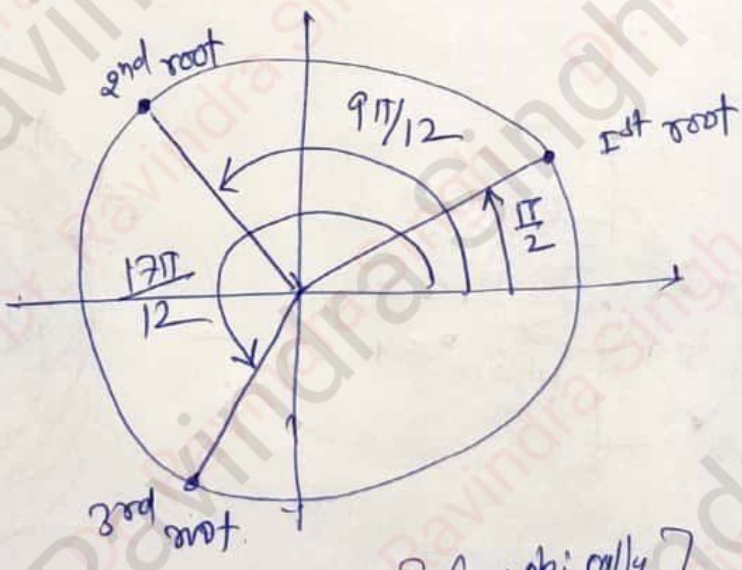
Sol:-
$$1+i = r(\cos\theta + i\sin\theta)$$
$$r = |1+i| = \sqrt{1+1} = \sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$$
$$\therefore\ 1+i = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] \text{ OR } \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4} \text{ OR } \sqrt{2}\,e^{i\pi/4}$$
$$\therefore\ (1+i)^{1/3} = \left\{\sqrt{2}\left[\cos \pi/4 + i\sin \pi/4\right]\right\}^{1/3}$$
$$= 2^{1/6}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi k}{3}\right) + i\sin\frac{1}{3}\left(\pi/4+2\pi k\right)\right]$$

If we put
$k = 0, 1, 2$ ... we get 1^{st}, 2^{nd} & 3^{rd} roots resp.

$k=0$, 1^{st} root (z_1) $= 2^{1/6}\left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]$

$k=1$, 2^{nd} root (z_2) $= 2^{1/6}\left[\cos\left(\frac{9\pi}{12}\right) + i\sin\left(\frac{9\pi}{12}\right)\right]$

$k=2$, 3^{rd} root (z_3) $= 2^{1/6}\left[\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right]$



[Graphically]

Logarithmic function of a complex variable

If $z = e^{\omega}$ where z & ω are complex numbers

then $\omega = \log z$ which is a multivalued function.

The general value of $\log z$ is defined by $\operatorname{Log} z$ where

$\boxed{\operatorname{Log} z = \log z + 2n\pi i}$ or $\boxed{\operatorname{Log} z = \log z + 2\pi k i}$

Separate $\log_e z$ into real & img parts,

where $z = x + iy$

Sol^n: $z = x + iy$

let $x = r\cos\theta$ --- (1)

$y = r\sin\theta$ --- (2)

from (1) & (2) squaring & adding we get

$r = \sqrt{x^2+y^2}$ --- (3)

also dividing (2) by (1) $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$ --- (4)

$\therefore$ ~~$\log(x+iy)$~~

$$\log_e z = \log_e(x+iy) = \log_e\{r(\cos\theta + i\sin\theta)\}$$

$$= \log_e r + \log_e\{(\cos(\theta+2n\pi) + \sin(\theta+2\pi n)\}$$

$$= \log_e r + \log_e e^{i(2n\pi+\theta)}$$

$$= \log_e r + i(2n\pi+\theta)$$

$$\boxed{\log_e z = \log_e\sqrt{x^2+y^2} + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right)}$$

$\Rightarrow$ for $n = 0$, we get principal value of z.

$$\log_e z = \log_e\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$= \log|z| + i\tan^{-1}\left(\frac{y}{x}\right)$$

Obtain the principal values of

(i) $(1+i)^{2-i}$

(ii) $(i)^{-i}$

(iii) $\log(\sqrt{3}-i)$

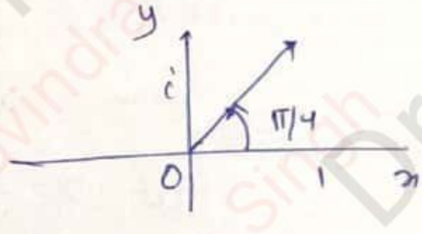
(iv) $(1+i)^{i}$

Sol^n: (iv) $(1+i)^{i} = e^{\log(1+i)^{i}} = e^{i\log(1+i)}$ ---(1)

Now $1+i = r(\cos\theta + i\sin\theta)$

$r = |1+i| = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$



$\therefore 1+i = \sqrt{2}\left[\cos\pi/4 + i\sin\pi/4\right]$

$= \sqrt{2}\left[\cos(\pi/4+2\pi k) + i\sin(\pi/4+2\pi k)\right]$

$= \sqrt{2}\, e^{i(\pi/4+2\pi k)}$

$\therefore \log(1+i) = \frac{1}{2}\log 2 + i(\pi/4+2\pi k)$ ---(2)

put (2) in (1)

$(1+i)^{i} = e^{i\left[\frac{1}{2}\log 2 + i(\pi/4+2\pi k)\right]}$

put $k=0$

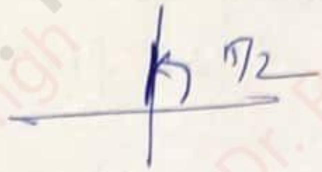
The principal value of $(1+i)^{i} = e^{-\pi/4} \cdot e^{\frac{i}{2}\log 2}$ Ans.

Note:-

(ii) i^{-i}

Sol:- $i^{-i} = e^{\frac{1}{i}\log i}$ — (1)

$i = r(\cos\theta + i\sin\theta)$



$r = 1, \quad \theta = \frac{\pi}{2}$

$\therefore i = (\cos\theta + i\sin\theta)$

$= [\cos(\pi/2 + 2\pi k) + i\sin(\pi/2 + 2\pi k)]$

$= e^{i(\pi/2 + 2\pi k)}$

$\log i = \log e^{i(\pi/2 + 2\pi k)}$

$= i(\pi/2 + 2\pi k)$ — (2)

put (2) in (1) we get

$i^{-i} = e^{\frac{1}{i}}\cdot(i(\pi/2 + 2\pi k))]$

$= e^{-i}[\cdot i(\pi/2 + 2\pi k)]$

put $k = 0$

$i^{-i} = e^{\pi/2}$. Ans It is the required principal value

Functions of a Complex Variable

$f(z)$ is a function of complex variable z and is denoted by w.

then $\boxed{w = f(z) = u(x,y) + i\,v(x,y)}$ - - (i)

where u & v are the real & img parts of $f(z)$

Limit of a function of a complex variable

Let $f(z)$ be a single valued function defined at all points in some nbhd. of point z_0. Then the limit of $f(z)$ as $z \to z_0$ is w_0.

$$\boxed{\lim_{z \to z_0} f(z) = w_0}$$

Continuity

$f(z)$ is said to be continuous at $z = z_0$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon$$

$$\& \ |z - z_0| < \delta$$

then $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Differentiability

Let $f(z)$ be a single valued function of variable z, then

$$\boxed{f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}}$$, provided the limit exists & indep. of path along which $\delta z \to 0$

Obtain the necessary & sufficient conditions for a function $f(z)$ to be analytic.

Sol^n :- Necessary Condition : Let $w = f(z) = u(x,y) + iv(x,y)$ be analytic in region R when u & v are the functions of x and y and Cauchy-Reimann equations are satisfied.

Proof:- Let Δu & Δv be the increments of u & v resp. corresponding to increments Δx and Δy of x and y.

Now
$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{(u+\Delta u) + i(v+\Delta v) - (u+iv)}{\Delta z}$$

$$= \frac{\Delta u + i\Delta v}{\Delta z} = \frac{\Delta u}{\Delta z} + i\frac{\Delta v}{\Delta z}$$

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0}\left(\frac{\Delta u}{\Delta z} + i\frac{\Delta v}{\Delta z}\right) \quad \text{--- (1)}$$

as $\Delta z \to 0$ along any path :

Along x-axis , , $z = x+iy$ ~~along x-axis~~ since $y = 0$

$$\left.\begin{array}{l} z = x \; ; \\ \Delta z = \Delta x \; , \; \Delta y = 0 \end{array}\right\} \text{put in (A)}$$

we get $f'(z) = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}\right) = \lim_{\Delta z \to 0}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \to$ (2)

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

also along y-axis, $z = x+iy = iy$ from $x = 0$

$$\therefore \Delta z = i\Delta y \quad \Big\} \text{(B)}$$

put (3) in (1)

$$f'(z) = \lim_{\Delta y \to 0}\left(\frac{\Delta u}{i\Delta y} + \frac{i\Delta v}{i\Delta y}\right) = \lim_{\Delta y \to 0}\left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{--- (3)}$$

If $f(z)$ is differentiable then (2) = (3)

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

equating real & img parts

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

& $\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$

These are called Cauchy-Reimann equations.

Sufficient Conditions:

Let $f(z)$ be a single valued function having $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$ at each point in R

then C-R eqns are satisfied.

By Taylor's theorem

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\delta x, y+\Delta y) \text{ will be}$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + \cdots i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right)\right]$$

$$= [u(x,y) + iv(x,y)] + \left[\frac{\partial u}{\partial x}\Delta x + i\frac{\partial v}{\partial x}\Delta x\right] + \left[\frac{\partial u}{\partial y}\Delta y + i\frac{\partial v}{\partial y}\Delta y + \cdots\right] + \cdots$$

$$= f(z) + \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\Delta x + \left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]\Delta y + \cdots \quad (1)$$

2nd & higher order terms ignored

also using C-R eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ in (1) we get}$$

$$f(z+\Delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\Delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\Delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\Delta x + \left[i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\Delta y$$

take two i common

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\left[\Delta x + i\Delta y\right]$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta z)$$

or

$$\boxed{\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

C-R Equations in Polar form

$$\boxed{\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta}\\ \frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r}\end{aligned}}$$

Proof. The polar form of a complex no. z is given by

$$z = r(\cos\theta + i\sin\theta)$$

$$z = re^{i\theta} \quad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$\therefore f(z) = f(re^{i\theta}) = u + iv \quad \text{---(1)}$$

differentiating (1) w.r.t r & θ we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\cdot e^{i\theta} \quad \text{---(2)}$$

$$\& \quad \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\cdot re^{i\theta}\cdot i \quad \text{---(3)}$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cdot r\cdot i \quad \text{using (2)}$$

$$= ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

equating real & imag. parts of above eqn. we get

$$\boxed{\begin{aligned}\frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r}\\ \& \quad \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta}\end{aligned}}$$

These are C-R equations in Polar form.

Analytic function :

A function $f(z) = u(x,y) + i\,v(x,y)$ is said to be analytic function of z if its real & img parts ie u & v satisfying C-R eqns./conditions

ie $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \&$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

OR

A function is said to be analytic at z_0 if it is differentiable at z_0 as well as every point of some nbhd. of z_0.

- Analytic function is also known as Holomorphic or Regular or monogenic
- $f(z)$ is analytic in a domain if it is analytic at every point of domain.

Singular Point :

The point at which function is not differentiable is called singular point of the function.

e.g the function $\frac{1}{z-2}$ has a singular point at $z=2$

Harmonic functions; If $f(z) = u(x,y) + i v(x,y)$ is an analytic function of z & C-R eqns are satisfied.

i.e.
$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \quad \& \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \quad \text{—} (1)$$

on differentiating (1)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \& \quad \text{—} (2)$$

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2} \quad \text{—} (3)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial y} \cdot \frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \nabla^2 u}$$

Similarly from (3)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 = \nabla^2 v$$

Thus
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \nabla^2 u$$
&
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 = \nabla^2 v$$

→ Laplace eqns. in 2-D.

or ~~which~~ solutions satisfied Laplace eqns are called Harmonic functions

$$\cancel{\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \cdot \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \cdot \frac{\partial u}{\partial y} = \frac{\partial^2 v}{\partial y^2}}$$

$$\cancel{\text{also } \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y} \cdot \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \cdot \frac{\partial v}{\partial y} = \frac{\partial^2 u}{\partial x^2}}$$

Q: Show that $f(z) = z^2$ is analytic in entire z plane

Sol^n:- let $f(z) = u+iv = z^2$

let $z = x+iy$

$\therefore z^2 = x^2-y^2+2xiy$

or $u+iv = x^2-y^2+2ixy$

$\therefore u = x^2-y^2, \quad v = 2xy$

Now $\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$

$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$

Now $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ &

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ Hence C-R eqns. are satisfied

So $f(z) = z^2$ is analytic in entire z-plane